

UNIQUENESS THEOREMS FOR (SUB-)HARMONIC FUNCTIONS WITH APPLICATIONS TO OPERATOR THEORY

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ABSTRACT. We obtain uniqueness theorems for harmonic and subharmonic functions of a new type. They lead to new analytic extension criteria and new conditions for stability of operator semigroups in Banach spaces with Fourier type.

1. INTRODUCTION

If u is a non-negative subharmonic function on the unit disc \mathbb{D} such that $\lim_{z \rightarrow \xi} u(z) = 0$ for every ξ on the unit circle \mathbb{T} , then the classical maximum principle forces u to be zero. In this article we shall present several generalisations of this uniqueness principle.

It is natural to ask whether the above uniqueness result remains true under weaker conditions on the boundary behaviour of u . For example, we may consider only nontangential or even radial boundary values. However, in this case the uniqueness principle is no longer true unless we impose some additional restrictions on u ; if $P(r, \varphi)$ is the Poisson kernel in \mathbb{D} , then the function

$$(1.1) \quad u(re^{i\varphi}) = \frac{\partial P(r, \varphi)}{\partial \varphi} = - \sum_{n=1}^{\infty} nr^n \sin n\varphi,$$

is harmonic in \mathbb{D} , $\lim_{r \rightarrow 1-} u(re^{i\varphi}) = 0$ for every $\varphi \in [0, 2\pi]$, but clearly we have $u \neq 0$, [26], [11], [2]. In the class of subharmonic functions for which

$$M_r(u) := \sup_{\varphi \in [0, 2\pi)} |u(re^{i\varphi})|, \quad r \in [0, 1),$$

grows sufficiently slowly as $r \rightarrow 1-$ it does suffice to consider only radial boundary values; the following radial uniqueness theorem was proved by B. Dahlberg, [11, Theorem 1].

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Theorem 1.1 (Dahlberg). *Let u be a non-negative subharmonic function on the unit disc. Assume that*

- (i) $\lim_{r \rightarrow 1-} u(re^{i\varphi}) = 0$ for every $\varphi \in [0, 2\pi)$;
- (ii) $M_r(u) = o((1-r)^{-2})$ as $r \rightarrow 1-$.

Then $u = 0$.

The second condition cannot be improved by replacing “o” by “O” as the example from (1.1) shows. Thus, Theorem 1.1 is optimal with respect to the growth conditions on $M_r(u)$.

If $u(z)$ tends to zero as $z \rightarrow \xi$ non-tangentially (that is, z stays in a sector with a fixed opening and vertex at ξ) for every $\xi \in \mathbb{T}$, then the growth condition on $M_r(u)$ in Theorem 1.1 can be relaxed, depending on the opening of the sector. Such sectorial variant of Theorem 1.1 was obtained by R. Berman and W. Cohn, [5, Theorem 1]. For $\theta \in (0, \frac{\pi}{2})$ and $\varphi \in [0, 2\pi)$, define the sector $\Omega_\theta(\varphi) := \{z \in \mathbb{D} : |\arg(1 - e^{-i\varphi}z)| \leq \frac{\pi}{2} - \theta\}$.

Theorem 1.2 (Berman-Cohn). *Let u be a non-negative subharmonic function on the unit disc, and let $\theta \in (0, \frac{\pi}{2})$ be fixed. Assume that*

- (i) $\lim_{\substack{z \rightarrow e^{i\varphi} \\ z \in \Omega_\theta(\varphi)}} u(z) = 0$ for every $\varphi \in [0, 2\pi)$;
- (ii) $M_r(u) = o((1-r)^{-\frac{\pi}{\theta}})$ as $r \rightarrow 1-$.

Then $u = 0$.

This statement is sharp in the sense that for every $\theta \in (0, \frac{\pi}{2})$ there exists a nonzero subharmonic function u such that $M_r(u) = O((1-r)^{-\frac{\pi}{\theta}})$ as $r \rightarrow 1-$, $\lim_{z \rightarrow \xi} u(z) = 0$ for every $\xi \in \mathbb{T} \setminus \{1\}$, and $u = 0$ in $\Omega_\theta(0)$; see [5, p. 283-286] for more details. Theorem 1.1 can be considered as the limit case of Theorem 1.2 with $\theta = \frac{\pi}{2}$.

One may ask whether Theorems 1.1 and 1.2 can be improved if u is a *harmonic* function. Since the function from (1.1) is harmonic, Theorem 1.1 is sharp also in the class of harmonic functions. We note that Theorem 1.1 for harmonic functions was proved by V. L. Shapiro [26] with a proof different from Dahlberg’s proof.

On the other hand, Theorem 1.2 can be improved for harmonic functions. The best results in this direction belong to F. Wolf; see [31] and in particular Theorem 7.4.1 and the proof of Theorem 7.4.6 therein.

Theorem 1.3 (Wolf). *Let u be a harmonic function on the unit disc, and let $\theta \in (0, \frac{\pi}{2})$ be fixed. Assume that*

- (i) $\lim_{\substack{z \rightarrow e^{i\varphi} \\ z \in \Omega_\theta(\varphi)}} u(z) = 0$ for every $\varphi \in [0, 2\pi)$;
- (ii) $M_r(u) = O(e^{\varepsilon(1-r)^{-\frac{\pi}{2\theta}}})$ as $r \rightarrow 1-$, for every $\varepsilon > 0$.

Then $u = 0$.

We modify an example of Rudin in [2, p. 39] to show that Theorem 1.3 is optimal with respect to the growth of $M_r(u)$, see Example 6.1.

Having in mind applications of uniqueness theorems to the study of spaces of functions with integral norm, such as e.g. Hardy spaces or Bergman spaces, one might wish to replace the sup-norm in the definition of $M_r(u)$ by the L^p -norm for some $p \geq 1$. Unfortunately, such a generalisation appears to be either impossible or superfluous. Indeed, as it was observed in [26] for the function u from (1.1), $\|u(re^{i\cdot})\|_p = O((1-r)^{-2+\frac{1}{p}}) = o((1-r)^{-2})$ as $r \rightarrow 1-$. On the other hand, if $\|u(re^{i\cdot})\|_p = o((1-r)^{-2+\frac{1}{p}})$ for some subharmonic function u , then $\|u(re^{i\cdot})\|_\infty = o((1-r)^{-2})$ as $r \rightarrow 1-$ by simple estimates of Poisson integrals, see [26]. So in this case, the conclusion $u = 0$ follows from Theorem 1.1. It is not clear what is a natural reformulation of Theorem 1.2 in the L^p -setting.

However, one can combine conditions from Theorem 1.2 with L^p -conditions by assuming that

$$|u| \leq f \cdot g \text{ in } \mathbb{D},$$

where f is a lower semicontinuous function satisfying condition (i) of Theorem 1.2, while g is measurable and $\sup_{r \in [0,1)} \|g(re^{i\cdot})\|_p < \infty$. If, in addition, $M_r(u)$ is of at most polynomial growth, then we are able to prove uniqueness theorems similar to the results stated above. This will be the main subject of the present paper. For example, we prove the following uniqueness theorem for harmonic functions.

Theorem 1.4 (Uniqueness principle). *Let u be a harmonic function on the unit disc. Assume that*

- (1) *there exist a lower semicontinuous function $f : \mathbb{D} \rightarrow \mathbb{R}_+$ and a measurable function $g : \mathbb{D} \rightarrow \mathbb{R}_+$ such that*
- (i) $|u(z)| \leq f(z)g(z)$ for every $z \in \mathbb{D}$,
 - (ii) $\lim_{\substack{z \rightarrow e^{i\varphi} \\ z \in \Omega_\theta(\varphi)}} f(z) = 0$ for every $\varphi \in [0, 2\pi)$ and some $\theta \in (0, \frac{\pi}{2})$,
 - (iii) $\sup_{r \in (0,1)} \int_0^{2\pi} g(re^{i\varphi}) d\varphi < \infty$,
- (2) *there exists $m \geq 0$ such that*

$$M_r(u) = O((1-r)^{-m}), \quad r \rightarrow 1-.$$

Then $u = 0$.

In this article we also obtain a version of Theorem 1.4 for non-negative subharmonic functions, as well as two versions of these theorems which deal only with the *local* boundary behaviour of harmonic and subharmonic functions. These main theorems as well as a generalized Phragmén-Lindelöf principle are stated in Section 2 and proved in Sections 3 and 4.

In Section 5 we present some applications of our main results. First, we apply our local theorems to the uniqueness theorem above, and to analytic continuation problems. After that, we obtain new optimal conditions for stability of operator semigroups on Banach spaces with Fourier type. In this application to operator semigroups conditions, using the resolvent identity, we obtain in a natural way conditions like those of Theorem 1.4.

Finally, in Section 6 we sketch an example that shows how sharp is Theorem 1.3.

Note that uniqueness theorems for harmonic functions are also related to the study of uniqueness problems for trigonometric series. We do not discuss such relations here and refer the interested reader to the papers [30, 31, 26, 17, 2].

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2. BOUNDARY BEHAVIOUR OF (SUB-)HARMONIC FUNCTIONS

To formulate our main result, we define the rectangle

$$R := \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1, 0 < \operatorname{Im} z < 1\},$$

and for every $\theta \in (0, \frac{\pi}{2})$ we define the sector

$$\Sigma_\theta := \{z \in \mathbb{C} : \theta < \arg z < \pi - \theta\}.$$

The statement below is a local halfplane version of the uniqueness Theorem 1.4.

Theorem 2.1. *Let u be a non-negative subharmonic function on R which extends continuously to $\bar{R} \setminus \mathbb{R}$. Assume that*

(1) *there exist a lower semicontinuous function $f : \bar{R} \setminus \mathbb{R} \rightarrow \mathbb{R}_+$ and a measurable function $g : R \rightarrow \mathbb{R}_+$ such that*

$$(i) \quad u(z) \leq f(z) g(z), \quad z \in R,$$

$$(ii) \quad \lim_{\substack{z \rightarrow \alpha \\ z \in \alpha + \Sigma_\theta}} f(z) = 0 \text{ for every } \alpha \in (-1, 1) \text{ and some } \theta > 0,$$

$$(iii) \quad \sup_{\beta \in (0,1)} \int_{-1}^1 g(\alpha + i\beta) d\alpha < \infty,$$

(2) *there exists $m \in [0, \frac{\pi}{\theta})$ such that*

$$\sup_{\alpha \in [-1,1]} u(\alpha + i\beta) = O(\beta^{-m}), \quad \beta \rightarrow 0+.$$

Then the function u admits a continuous extension to the interval $(-1, 1)$, and $u = 0$ on $(-1, 1)$.

Remark 2.2. Note that Σ_θ has an angular opening $\pi - 2\theta$. Thus, Theorem 2.1 agrees with Theorem 1.2 as far as the relation between θ and m is concerned. It is not clear whether in condition (2) one can replace the $O(\beta^{-m})$ for some $m \in [0, \frac{\pi}{\theta})$ by just $o(\beta^{-\frac{\pi}{\theta}})$. By the argument from [5, p.283-286] we cannot write $O(\beta^{-\frac{\pi}{\theta}})$, even for $g = 1$.

For harmonic functions, the above result can be improved in the sense that in the second condition the constant m does not depend on the angle θ from the first condition.

Theorem 2.3. *Let $u : R \rightarrow \mathbb{C}$ be a harmonic function which extends continuously to $\bar{R} \setminus \mathbb{R}$. Assume that the condition (1) of Theorem 2.1 holds, and that, in addition,*

(2) *there exists $m \geq 0$ such that*

$$(2.1) \quad \sup_{\alpha \in [-1,1]} |u(\alpha + i\beta)| = O(\beta^{-m}), \quad \beta \rightarrow 0+.$$

Then the function u admits a continuous extension to the interval $(-1, 1)$, and $u = 0$ on $(-1, 1)$.

Clearly, the condition (2) of Theorem 2.3 is weaker than the corresponding condition from Theorem 2.1, so that we obtain a stronger result for harmonic functions. The proof of Theorem 2.3 is similar to the proof of Theorem 2.1 apart from one step. In this step, we need a proposition of Phragmén-Lindelöf type which we state separately.

Proposition 2.4. *Let $u : \Delta_\rho \rightarrow \mathbb{R}$ be a harmonic function on the finite open sector*

$$\Delta_\rho := \{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{2}, |z| < \rho\}.$$

Assume that

- (1) *u has a continuous extension to $\overline{\Delta_\rho} \setminus \{0\}$,*
- (2) *there exist $l \in [0, 2)$, $\theta \in (0, \frac{\pi}{2})$ such that $\frac{\theta}{\pi} \notin \mathbb{Q}$ and*

$$\sup_{\substack{z \in \overline{\Delta_\rho} \setminus \{0\} \\ \arg z \in \{0, \theta, \frac{\pi}{2}\}}} |z|^l |u(z)| < \infty,$$

- (3) *there exists $m \in [0, \infty)$ such that*

$$\sup_{z \in \Delta_\rho} |\operatorname{Im} z|^m |u(z)| < \infty.$$

Then

$$\sup_{z \in \Delta_\rho} |z|^l |u(z)| < \infty.$$

Remark 2.5. Classical Phragmén-Lindelöf principle does not directly apply to the situation of Theorem 2.4 since there is no relation between the numbers θ and m in the assumptions (2) and (3).

3. PROOF OF THEOREMS 2.1 AND 2.3

In what follows, we denote by C a positive constant which may vary from line to line.

We start by stating two lemmas which are needed for the proofs of Theorems 2.1, 2.3 and Proposition 2.4.

The first lemma, a version of the mean value inequality for subharmonic functions, can be found in KOOSIS [20, Lemma, Section VIII D.2] or GARNETT [15, Chapter III, Lemma 3.7].

Lemma 3.1. *Let u be a non-negative subharmonic function defined on a domain $D \subset \mathbb{C}$ and let $p > 0$. Then for every $z \in D$ and every closed ball $B(z, r) \subset D$ we have*

$$(3.1) \quad u(z)^p \leq C_p \frac{1}{|B(z, r)|} \int_{B(z, r)} u(y)^p dy,$$

where C_p is a constant depending only on p .

The second lemma is a variant of a result by DOMAR [12]. Our proof uses an idea from [2, Lemma 4].

Lemma 3.2. *Let $u : U \setminus \{0\} \rightarrow \mathbb{R}_+$ be a subharmonic function, where U is a neighbourhood of the closed unit disc \mathbb{D} . Assume that for some constants $C, m \geq 0$ and for all $z \in \mathbb{D} \setminus \{0\}$*

$$(3.2) \quad u(z) \leq C |\operatorname{Im} z|^{-m}.$$

Then there is another constant $C > 0$ such that for all $z \in \mathbb{D} \setminus \{0\}$

$$u(z) \leq C |z|^{-m}.$$

Proof. Let $z \in \mathbb{D}$ be fixed, and let $r := r(z) := |z|$.

If $|\operatorname{Im} z| \geq r/2$, then the above inequality is a direct consequence of assumption (3.2).

So we can assume that $|\operatorname{Im} z| \leq r/2$. Choose $\delta > 0$ independent of $z \in \mathbb{D} \setminus \{0\}$ such that the closed disc $\bar{B}(z, \delta|z|)$ is contained in $U \setminus \{0\}$. Applying the mean value inequality (3.1) with $p = \frac{1}{2m}$ (Lemma 3.1) and using the assumption (3.2) we obtain

$$\begin{aligned} u(z)^{\frac{1}{2m}} &\leq C \frac{1}{\delta^2 r^2 \pi} \int_{B(z, \delta r)} u(y)^{\frac{1}{2m}} dy \\ &\leq \frac{C}{r^2} \int_{B(z, \delta r)} |\operatorname{Im} y|^{\frac{1}{2}} dy \\ &\leq \frac{C}{r^2} \int_{\operatorname{Re} z - \delta r}^{\operatorname{Re} z + \delta r} \int_{\operatorname{Im} z - \delta r}^{\operatorname{Im} z + \delta r} |\beta|^{-\frac{1}{2}} d\beta d\alpha \\ &\leq \frac{C}{r} \left(|\operatorname{Im} z + \delta r|^{\frac{1}{2}} + |\operatorname{Im} z - \delta r|^{\frac{1}{2}} \right) \\ &\leq C r^{-\frac{1}{2}} = C |z|^{-\frac{1}{2}}. \end{aligned}$$

This is the claim. \square

For the proof of Theorem 2.1 we also need the following Phragmén-Lindelöf principle for functions which are subharmonic in a finite sector. Similar (and more general) statements for infinite sectors can be found in LEVIN [22, Theorem 3, p. 49] or HARDY & ROGOSINSKI [16]. The proof of our statement is an easy adaptation of their proofs.

Proposition 3.3 (Classical Phragmén-Lindelöf principle). *Let u be a non-negative subharmonic function on the open finite sector*

$$\Delta_{\theta, r} := \{z \in \mathbb{C} : |\arg z| < \theta, |z| < r\}, \quad \theta \in (0, \pi), r > 0.$$

Assume that

- (1) u has a lower semicontinuous extension to $\overline{\Delta_{\theta,r}} \setminus \{0\}$,
(2) there exist $0 \leq l \leq m < \frac{\pi}{2\theta}$ such that

$$\sup_{\substack{z \in \Delta_{\theta,r} \\ |\arg z| = \theta}} |z|^l u(z) < \infty,$$

and

$$\sup_{z \in \Delta_{\theta,r}} |z|^m u(z) < \infty.$$

Then

$$\sup_{z \in \Delta_{\theta,r}} |z|^l u(z) < \infty.$$

Proof of Theorem 2.1. We call a point $\alpha \in (-1, 1)$ *regular* if u has a continuous extension to a neighbourhood of α in \bar{R} and we call it *singular* otherwise. We denote the set of all singular points in $(-1, 1)$ by S . Clearly, S is closed in $(-1, 1)$.

Step 1: We first prove that $u = 0$ on $(-1, 1) \setminus S$. Otherwise, by continuity, we can find a nonempty interval $U \subset (-1, 1) \setminus S$ such that $u(\alpha) \geq c > 0$ for every $\alpha \in U$. By assumption (1) (ii) and Egorov's theorem, we find a set $V \subset U$ of positive measure such that

$$\lim_{\beta \rightarrow 0+} \sup_{\alpha \in V} f(\alpha + i\beta) = 0.$$

Together with assumptions (1) (i), (iii) and Fatou's lemma this implies

$$\begin{aligned} 0 &< \liminf_{\beta \rightarrow 0+} \int_V u(\alpha + i\beta) d\alpha \\ &\leq \limsup_{\beta \rightarrow 0+} \int_V f(\alpha + i\beta) g(\alpha + i\beta) d\alpha = 0, \end{aligned}$$

a contradiction. Hence, u vanishes at every regular point and it remains to prove that the set S of singular points in $(-1, 1)$ is empty.

We assume in the following that S is nonempty and we will show that this leads to a contradiction.

Step 2: We define for every $n \in \mathbb{N}$

$$\begin{aligned} S_n &:= \{\alpha \in S : \sup_{z \in R \cap \alpha + \Sigma_\theta} f(z) \leq n\} \\ &= \bigcap_{\beta > 0} \{\alpha \in S : \sup_{\substack{z \in R \cap \alpha + \Sigma_\theta \\ \operatorname{Im} z \geq \beta}} f(z) \leq n\}. \end{aligned}$$

By lower semicontinuity of the function f , the sets S_n are closed. By assumption (1) (ii),

$$S = \bigcup_{n \in \mathbb{N}} S_n.$$

By Baire's category theorem, there exists $n \in \mathbb{N}$ such that S_n has nonempty interior in S , i.e. there exists a nonempty interval $(a, b) \subset (-1, 1)$ such that

$$(3.3) \quad S \cap (a, b) = S_n \cap (a, b) \neq \emptyset.$$

Without loss of generality we can assume that $a > -1$ and $b < 1$. Since the set $(a, b) \setminus S$ is open, it is the countable union of mutually disjoint intervals $I_k := (a_k, b_k)$,

$$(a, b) \setminus S = \bigcup_k (a_k, b_k),$$

In the following, we put for every k

$$\begin{aligned} l_k &:= b_k - a_k = |I_k| & \text{and} \\ h_k &:= l_k \frac{\tan \theta}{2}, \end{aligned}$$

and we also define

$$h := \min\{(a+1)\tan\theta, (1-b)\tan\theta, 1\} > 0.$$

Furthermore, we consider the rectangles

$$\tilde{R} := \{z \in R : a < \operatorname{Re} z < b\}$$

and

$$R_k := \{z \in R : a_k < \operatorname{Re} z < b_k\}.$$

The parameter h is only of technical interest: it will be convenient to know that for every $\alpha \in [a, b]$ one has

$$\{z \in \mathbb{C} : z \in \alpha + \Sigma_\theta, \operatorname{Im} z < h\} \subset R.$$

Step 3: We prove that

$$(3.4) \quad \sup_{0 < \beta < 1} \int_a^b u(\alpha + i\beta) d\alpha < \infty.$$

Step 3.1: It follows from (3.3) that for every $\beta \in (0, 1)$

$$\int_{(a,b) \cap S} u(\alpha + i\beta) d\alpha \leq n \int_{(a,b) \cap S} g(\alpha + i\beta) d\alpha \leq nC,$$

so that

$$(3.5) \quad \sup_{0 < \beta < 1} \int_{(a,b) \cap S} u(\alpha + i\beta) d\alpha < \infty.$$

Step 3.2: Fix $\beta \in (0, 1)$. If $\beta \geq h_k$, then

$$I_k + i\beta \subset (S \cap (a, b)) + \Sigma_\theta,$$

so that

$$\begin{aligned} \sum_{\substack{k \\ \beta \geq h_k}} \int_{a_k}^{b_k} u(\alpha + i\beta) d\alpha &\leq \sum_{\substack{k \\ \beta \geq h_k}} \int_{a_k}^{b_k} n g(\alpha + i\beta) d\alpha \\ &\leq n \sup_{0 < \beta < 1} \int_a^b g(\alpha + i\beta) d\alpha \leq nC. \end{aligned}$$

Hence,

$$(3.6) \quad \sup_{0 < \beta < 1} \sum_{\substack{k \\ \beta \geq h_k}} \int_{a_k}^{b_k} u(\alpha + i\beta) d\alpha < \infty.$$

Step 3.3: We prove that

$$(3.7) \quad \sup_{0 < \beta < 1} \sum_{\substack{k \\ \beta < h_k}} \int_{a_k}^{b_k} u(\alpha + i\beta) d\alpha < \infty.$$

In what follows, if $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, is a simply connected domain then we denote by ω_D the harmonic measure associated with D ; see e.g. [21]. If $h : \partial D \rightarrow \mathbb{R}_+$ is a Borel function such that $\int_{\partial D} |h(\xi)| d\omega_D(\xi, z) < \infty$ for every $z \in D$, then we can define

$$(3.8) \quad P_D(h)(z) := \int_{\partial D} h(\xi) d\omega_D(\xi, z), \quad z \in D.$$

The function $P_D(h)$ is harmonic in D , and if h is continuous and bounded, then $P_D(h)$ is actually the unique continuous solution of the Dirichlet problem on \bar{D} with boundary value h .

If ∂D is piecewise smooth, and $\varphi : D \mapsto \mathbb{D}$ is a conformal mapping, then (3.8) can be rewritten as

$$(3.9) \quad P_D(h)(\varphi^{-1}(z)) = \frac{1}{2\pi} \int_{\partial \mathbb{D}} h(\varphi^{-1}(\xi)) \frac{1 - |z|^2}{|z - \xi|^2} |d\xi|.$$

The next technical result will be used several times in the sequel.

Lemma 3.4. *Let $D \subset \mathbb{C}$ be a piecewise smooth, bounded, simply connected domain such that ∂D has a finite set of corners C . Assume that the opening of each corner is $\pi\alpha$, $\alpha \in [\frac{1}{2}, 1)$. Let $0 < \alpha\beta < 1$, and let $h : \partial D \rightarrow \mathbb{C}$ be a function which is continuous on $\partial D \setminus C$ and which satisfies*

$$|h(\xi)| = O(|\xi - c|^{-\beta}), \quad \text{as } \xi \rightarrow c, \xi \in \partial D,$$

for each $c \in C$. Then the following statements are true:

- (i) for every $z \in D$ one has $h \in L^1(\partial D, d\omega_D(\cdot, z))$;
- (ii) if h is continuous at $\xi \in \partial D$, then $\lim_{z \rightarrow \xi} P_D(h)(z) = h(\xi)$;
- (iii) $|P_D(h)(z)| = O(|z - \xi|^{-\beta})$ as $z \rightarrow \xi$, $\xi \in C$.

Lemma 3.4 follows from (3.9) and the facts that φ extends continuously to \bar{D} , satisfies $\varphi(\xi) = O(|\xi - c|^{1/\alpha})$ as $\xi \rightarrow c$, for every $c \in C$, and φ^{-1} extends continuously to $\bar{\mathbb{D}}$.

Step 3.3.1: We show that for every k

$$(3.10) \quad u(z) \leq C(k) \max \left\{ \frac{1}{|z - a_k|}, \frac{1}{|z - b_k|} \right\} \text{ for every } z \in R_k.$$

Fix k , let $\theta_1 \in (\theta, \frac{\pi}{2})$ and choose $\delta > 0$ so small that

$$z \in \Sigma_{\theta_1} \quad \Rightarrow \quad B(z, |z|\delta) \subset \Sigma_{\theta}.$$

Then for every $z \in \Sigma_{\theta_1}$ with $|z|$ small enough the mean value inequality implies

$$\begin{aligned} u(a_k + z) &\leq \frac{C}{|z|^2} \int_{B(a_k + z, |z|\delta)} u(z') \, dz' \\ &\leq \frac{Cn}{|z|^2} \int_{B(a_k + z, |z|\delta)} g(z') \, dz' \\ &\leq \frac{Cn}{|z|^2} \int_{\operatorname{Im} z - |z|\delta}^{\operatorname{Im} z + |z|\delta} \int_a^b g(\alpha + i\beta) \, d\alpha \, d\beta \\ &\leq \frac{C}{|z|}, \end{aligned}$$

for some constant C which depends only on δ (i.e. θ_1 and θ), n and g . Similarly, for every $z \in \Sigma_{\theta_1}$ with $|z|$ small enough,

$$u(b_k + z) \leq \frac{C}{|z|}.$$

Now we choose θ_1 sufficiently close to θ so that $m < \frac{\pi}{\theta_1}$ (where m is as in assumption (2)). The Phragmén-Lindelöf principle from Proposition 3.3 implies (3.10).

Step 3.3.2: By (3.10) and Lemma 3.4, for every k and every $z \in R_k$ the function $u|_{\partial R_k}$ is integrable with respect to the harmonic measure $\omega_{R_k}(\cdot, z)$. We show that for every k

$$(3.11) \quad u(z) \leq P_{R_k}(u)(z) \text{ for every } z \in R_k.$$

Clearly, the function $v_k(z) := u(z) - P_{R_k}(u)(z)$ ($z \in R_k$) is subharmonic in R_k , continuous up to $\overline{R_k} \setminus \{a_k, b_k\}$, and $v_k = 0$ on $\partial R_k \setminus \{a_k, b_k\}$. Let $R_{k,\varepsilon} := \{z \in R_k : \operatorname{Im} z > \varepsilon\}$. Then, for every $z \in R_k$ and every $\varepsilon \in (0, \operatorname{Im} z)$,

$$v_k(z) \leq \int_{\partial R_{k,\varepsilon}} v_k(\xi) d\omega_{R_{k,\varepsilon}}(\xi, z).$$

Letting $\varepsilon \rightarrow 0+$, by (3.10) and the dominated convergence theorem, we obtain

$$v_k(z) \leq \int_{\partial R_k} v_k(\xi) d\omega_{R_k}(\xi, z) = 0.$$

This implies (3.11).

Step 3.3.3: By (3.11),

$$(3.12) \quad \sum_{\substack{k \\ \beta < h_k}} \int_{a_k}^{b_k} u(\alpha + i\beta) d\alpha \leq \sum_{\substack{k \\ \beta < h_k}} \left\{ \int_{a_k}^{b_k} P_{R_k}(u \chi_{L_k})(\alpha + i\beta) d\alpha + \right. \\ \left. + \int_{a_k}^{b_k} P_{R_k}(u \chi_{M_k})(\alpha + i\beta) d\alpha + \right. \\ \left. + \int_{a_k}^{b_k} P_{R_k}(u \chi_{M'_k})(\alpha + i\beta) d\alpha \right\},$$

where

$$\begin{aligned} L_k &:= \{z \in \partial R_k : \operatorname{Im} z = 1\}, \\ M_k &:= \{z \in \partial R_k : \operatorname{Re} z = a_k\}, \text{ and} \\ M'_k &:= \{z \in \partial R_k : \operatorname{Re} z = b_k\}. \end{aligned}$$

By continuity, we have

$$\sup_{\alpha \in [-1, 1]} u(\alpha + i) = C < \infty,$$

and

$$P_{R_k}(u \chi_{L_k})(z) \leq C \text{ for every } z \in R_k \text{ and every } k.$$

Hence,

$$\sum_{\substack{k \\ \beta < h_k}} \int_{a_k}^{b_k} P_{R_k}(u \chi_{L_k})(\alpha + i\beta) d\alpha \leq \sum_{\substack{k \\ \beta < h_k}} \int_{a_k}^{b_k} C d\alpha \leq C(b - a),$$

so that we have estimated the first term on the right-hand side of (3.12) by a constant independent of $\beta \in (0, h_k)$.

Let

$$Q := \left\{ z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{2} \right\}.$$

For every k we define $u_k : \partial Q \rightarrow \mathbb{R}_+$ by

$$u_k(i\beta) := \begin{cases} u(a_k + i\beta) & \text{if } 0 < \beta \leq 1, \\ 0 & \text{if } \beta > 1, \end{cases} \quad \text{and } u_k(\alpha) = 0, \quad \alpha > 0.$$

Fix $\beta > 0$ and put $r = \beta \cos \theta$. Then one checks that the intervals $((a_k - r, a_k + r))_{\substack{k \\ \beta < h_k}}$ are mutually disjoint. This fact, the mean value inequality, and the assumption (1) imply that

$$\begin{aligned} \sum_{\substack{k \\ \beta < h_k}} u_k(i\beta) &= \sum_{\substack{k \\ \beta < h_k}} u(a_k + i\beta) \\ &\leq \frac{C}{r^2} \sum_{\substack{k \\ \beta < h_k}} \int_{|z - a_k - i\beta| \leq r} u(z) \, dz \\ &\leq \frac{Cn}{r^2} \sum_{\substack{k \\ \beta < h_k}} \int_{\beta-r}^{\beta+r} \int_{a_k-r}^{a_k+r} g(\alpha' + i\beta') \, d\alpha' \, d\beta' \\ &\leq \frac{C}{r^2} \int_{\beta-r}^{\beta+r} \int_a^b g(\alpha' + i\beta') \, d\alpha' \, d\beta' \\ &\leq \frac{C}{r} \leq \frac{C}{\beta}. \end{aligned}$$

We deduce that

$$\begin{aligned} \sum_{\substack{k \\ \beta < h_k}} \int_{a_k}^{b_k} P_{R_k}(u \chi_{S_k})(\alpha + i\beta) \, d\alpha &\leq \sum_{\substack{k \\ \beta < h_k}} \int_0^\infty P_Q(u_k)(\alpha + i\beta) \, d\alpha \\ &= \int_0^\infty P_Q\left(\sum_{\substack{k \\ \beta < h_k}} u_k\right)(\alpha + i\beta) \, d\alpha \\ &\leq C \int_0^\infty P_Q(v)(\alpha + i\beta) \, d\alpha, \end{aligned}$$

where $v : \partial Q \rightarrow \mathbb{R}_+$ is defined by

$$v(i\beta') := \frac{1}{\beta'} \text{ and } v(\alpha') := 0, \quad \alpha', \beta' > 0.$$

Note that

$$P_Q(v)(z) = P(z), \quad z \in Q,$$

where P is the Poisson kernel in the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$. Indeed, if $v_0 = |P_Q(v) - P|$, then v_0 is subharmonic, continuous in $\overline{Q} \setminus \{0\}$, and, by Lemma 3.4 (iii), $|v_0(z)| = O(|z|^{-1})$

as $z \rightarrow 0$. By Proposition 3.3, v_0 is bounded, and by the classical maximum principle we conclude that $v_0 = 0$.

Since

$$\int_{\mathbb{R}} P(\alpha + i\beta) d\alpha = 1 \text{ for every } \beta > 0,$$

we have proved that the second term on the right-hand side of (3.12) can be estimated by a constant independent of $\beta \in (0, 1)$. The same argument works for the third term on the right-hand side of (3.12).

Hence, the right-hand side of (3.12) is uniformly bounded in $\beta \in (0, 1)$, and this proves (3.7).

Summing up (3.5), (3.6) and (3.7), we obtain (3.4).

Step 4: By (3.4) and the weak* sequential compactness of the unit ball of $M([a, b]) = C([a, b])^*$, there exists a sequence $(\beta_k) \searrow 0, k \rightarrow \infty$, and a finite positive Borel measure $\mu \in M([a, b])$ such that

$$w^* - \lim_{k \rightarrow \infty} u(\cdot + i\beta_k) = \mu \text{ in } M([a, b]).$$

Let $P_{\mathbb{C}_+}(\mu)$ be the Poisson integral of μ in the upper half-plane \mathbb{C}_+ .

Let $L := \partial\tilde{R} \cap \mathbb{C}_+$ be the part of the boundary of \tilde{R} which does not lie on the real axis. Since

$$u(z) \leq C \max \left\{ \frac{1}{|z - a|}, \frac{1}{|z - b|} \right\}, \quad z \in L,$$

by Lemma 3.4, the function $P_{\tilde{R}}(u \chi_L)$ is well defined and harmonic in \tilde{R} . Moreover,

$$\lim_{\beta \rightarrow 0+} P_{\tilde{R}}(u \chi_L)(\cdot + i\beta) = 0 \text{ locally uniformly on } (a, b).$$

Hence, if we define $v := \max((u - P_{\tilde{R}}(u \chi_L)), 0)$, then v is a non-negative subharmonic function on \tilde{R} , continuous up to L , and $v = 0$ on L . Moreover, for every continuous φ with support in (a, b) ,

$$(3.13) \quad \lim_{k \rightarrow \infty} \int_a^b v(\alpha + i\beta_k) \varphi(\alpha) d\alpha = \int_a^b \varphi(\alpha) d\mu.$$

On the other hand, for every such φ we also have

$$(3.14) \quad \lim_{\beta \rightarrow 0+} \int_a^b P_{\mathbb{C}_+}(\mu)(\alpha + i\beta) \varphi(\alpha) d\alpha = \int_a^b \varphi(\alpha) d\mu.$$

Now, given $\beta > 0$, denote $\tilde{R}_\beta := i\beta + \tilde{R}$. For $z \in \tilde{R}$,

$$\begin{aligned} P_{\mathbb{C}_+}(\mu)(z) &= \lim_{\beta \rightarrow 0+} P_{\mathbb{C}_+}(\mu)(z + i\beta) \\ &\geq \limsup_{\beta \rightarrow 0+} \int_a^b P_{\mathbb{C}_+}(\mu)(\alpha + i\beta) d\omega_{\tilde{R}_\beta}(\alpha, z + i\beta) \\ &= \limsup_{\beta \rightarrow 0+} \int_a^b P_{\mathbb{C}_+}(\mu)(\alpha + i\beta) d\omega_{\tilde{R}}(\alpha, z). \end{aligned}$$

Furthermore, since the function

$$\alpha \mapsto \frac{d\omega_{\tilde{R}}(\alpha, z)}{d\alpha}$$

is continuous on $[a, b]$ and vanishes at a and b , and since

$$\int_a^b P_{\mathbb{C}_+}(\mu)(\alpha + i\beta) d\alpha \leq C \text{ for every } \beta > 0,$$

we obtain by (3.14) that

$$P_{\mathbb{C}_+}(\mu)(z) \geq \limsup_{\beta \rightarrow 0+} \int_a^b \frac{d\omega_{\tilde{R}}(\alpha, z)}{d\alpha} d\mu(\alpha).$$

On the other hand, if we set $R_{\beta_k}^* := \{z \in \tilde{R} : \operatorname{Im} z > \beta_k\}$, then

$$\begin{aligned} v(z) &= \lim_{k \rightarrow \infty} v(z + i\beta_k) \leq \liminf_{k \rightarrow \infty} \int_a^b v(\alpha + i\beta_k) d\omega_{R_{\beta_k}^*}(\alpha, z + \beta_k) \\ &\leq \liminf_{k \rightarrow \infty} \int_a^b v(\alpha + i\beta_k) d\omega_{\tilde{R}_{\beta_k}}(\alpha, z + \beta_k) \\ &= \liminf_{k \rightarrow \infty} \int_a^b v(\alpha + i\beta_k) d\omega_{\tilde{R}}(\alpha, z). \end{aligned}$$

As above, we conclude by (3.4) and by (3.13) that

$$v(z) \leq \liminf_{k \rightarrow \infty} \int_a^b \frac{d\omega_{\tilde{R}}(\alpha, z)}{d\alpha} d\mu(\alpha).$$

Thus,

$$(3.15) \quad P_{\mathbb{C}_+}(\mu)(z) \geq v(z) \text{ for every } z \in \tilde{R},$$

and it follows from (3.13) and (3.14) that for every $a < a' < b' < b$

$$(3.16) \quad \lim_{k \rightarrow \infty} \int_{a'}^{b'} |(P_{\mathbb{C}_+}(\mu) - v)(\alpha + i\beta_k)| d\alpha = 0.$$

Assume that $\mu((a, b)) \neq 0$. Then one finds $a < a' < b' < b$ such that $\mu((a', b')) > 0$. By assumption (1) (ii) and Egorov's theorem, there exists a set $T \subset (a', b')$ of positive μ -measure such that

$$\lim_{\beta \rightarrow 0+} \sup_{\substack{\alpha \in (-1, 1) \\ \alpha + i\beta \in T + \Sigma_\theta}} f(\alpha + i\beta) = 0.$$

Together with assumption (1) (i) and (iii) this implies

$$\limsup_{\beta \rightarrow 0+} \int_{\substack{\alpha \in (-1, 1) \\ \alpha + i\beta \in T + \Sigma_\theta}} u(\alpha + i\beta) d\alpha = 0,$$

and thus also

$$(3.17) \quad \limsup_{\beta \rightarrow 0+} \int_{\substack{\alpha \in (-1, 1) \\ \alpha + i\beta \in T + \Sigma_\theta}} v(\alpha + i\beta) d\alpha = 0.$$

On the other hand, one easily checks that

$$\int_{\substack{\alpha \in (-1, 1) \\ \alpha + i\beta \in T + \Sigma_\theta}} P_{C+}(\mu)(\alpha + i\beta) d\alpha \geq C(\theta)\mu(T) > 0$$

for every $\beta > 0$. This inequality contradicts (3.16) and (3.17), and therefore the assumption $\mu((a, b)) \neq 0$ is wrong.

However, if $\mu((a, b)) = 0$, then by (3.15), $v = 0$ on \tilde{R} , and u is continuous on $\tilde{R} \cup (a, b)$. Thus,

$$S \cap (a, b) = \emptyset,$$

which contradicts to (3.3).

Hence, the assumption that S is nonempty is wrong. Therefore, S is empty and the theorem is proved. \square

Proof of Theorem 2.3. The proof of Theorem 2.1 goes through for the non-negative subharmonic function $|u|$, except for the inequality

$$(3.18) \quad |u(z)| \leq P_{R_k}(|u|)(z) \text{ for every } z \in R_k$$

from Step 3.3.2 that needs a justification.

As in the Step 3.3.1 of the proof of Theorem 2.1 one proves that for every $\theta_1 \in (\theta, \frac{\pi}{2})$ there exists a constant $C \geq 0$ such that for every $z \in \Sigma_{\theta_1}$ with $|z|$ small enough one has

$$|u(a_k + z)| \leq \frac{C}{|z|} \text{ and } |u(b_k + z)| \leq \frac{C}{|z|}.$$

By Proposition 2.4, there exists a constant $C = C(k) \geq 0$ such that for every $z \in R_k$,

$$|u(z)| \leq C \max \left\{ \frac{1}{|z - a_k|}, \frac{1}{|z - b_k|} \right\}.$$

Arguing as in Step 3.3.2, we obtain (3.18) for harmonic functions satisfying the weaker growth condition (2.1). The rest of the proof is the same as in Theorem 2.1. \square

4. PROOF OF PROPOSITION 2.4

Proof of Proposition 2.4. Without loss of generality, we may in the following assume that m is an integer larger than l .

Step 1: By assumptions (1)-(2) and Lemma 3.4 (i), the function $u|_{\partial\Delta_\rho}$ is integrable with respect to the harmonic measure ω_{Δ_ρ} associated with the sector Δ_ρ . Therefore, the function

$$v := P_{\Delta_\rho}(u|_{\partial\Delta_\rho}) = \int_{\partial\Delta_\rho} u(\xi) d\omega_{\Delta_\rho}(\xi, z), \quad z \in \Delta_\rho,$$

is well-defined. Moreover, by the assumption (1) and Lemma 3.4 (ii), v is continuous up to $\overline{\Delta_\rho} \setminus \{0\}$ and, by the assumption (2) and Lemma 3.4 (iii), there exists a constant $C \geq 0$ such that

$$(4.1) \quad |v(z)| \leq C|z|^{-l} \text{ for all } z \in \Delta_\rho.$$

Hence, it suffices to show that $u = v$.

Step 2: Let $w := u - v$. We define the punctured disc

$$D_\rho := \{z \in \mathbb{C} : |z| < \rho\} \setminus \{0\},$$

and we denote also by w the harmonic function on D_ρ which one obtains by reflecting the original w first at the real axis (using the Schwartz reflection principle) and then at the imaginary axis (using the Schwartz reflection principle again).

We prove that there exists a constant $C > 0$ such that

$$(4.2) \quad |w(z)| \leq C|z|^{-m} \text{ for all } z \in D_\rho.$$

In fact, by assumption (3) and the estimate (4.1), there exists a constant $C \geq 0$ such that

$$|w(z)| \leq C|\operatorname{Im} z|^{-m}, \quad z \in R.$$

The estimate (4.2) follows then from a simple application of Lemma 3.2.

Step 3: Since 0 is an isolated singularity of the harmonic function w , we have

$$w(re^{i\phi}) = a \log r + \sum_{n \in \mathbb{Z}} (r^n a_n \cos n\phi + r^n b_n \sin n\phi)$$

for all $z = re^{i\varphi} \in D_\rho$. The series converges absolutely on every compact subset of D_ρ . The estimate (4.2) implies that

$$(4.3) \quad w(re^{i\phi}) = a \log r + \sum_{-m \leq n \leq -1} (a_n r^n \cos n\phi + b_n r^n \sin n\phi) + w_0(re^{i\phi}),$$

with w_0 bounded in D_ρ .

By assumption (2), the function $r \rightarrow r^l w(re^{i\theta})$ is bounded on $(0, 1)$, and, moreover, $\cos n\theta$ and $\sin n\theta$, $-m \leq n \leq -1$, are nonzero. This implies that $a_n = b_n = 0$, $-m \leq n \leq -2$. Now recall that $w = 0$ on $(0, \rho)$ and on $(0, i\rho)$. Therefore, letting $\phi = 0$ in the representation (4.3), we conclude that $a = 0$ and $a_{-1} = 0$. Finally, if we let $\phi = \frac{\pi}{2}$ in (4.3), we obtain that $b_{-1} = 0$.

Hence, $w = w_0$ is bounded. The classical maximum principle (note that $w = 0$ on ∂D_r) implies that $w = 0$, i.e. $u = v$. \square

5. APPLICATIONS

5.1. Uniqueness principle. As a first application we prove the uniqueness principle for harmonic functions formulated in the Introduction.

Proof of Theorem 1.4. Let

$$\varphi_\theta(z) = \exp[2\pi i(z + \theta)], \quad z \in \mathbb{C}, \quad 0 \leq \theta < 1.$$

This holomorphic function φ_θ maps the interior of the rectangle

$$R := \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1, \quad 0 < \operatorname{Im} z < 1\}$$

onto the set

$$\{z \in \mathbb{C} : e^{-2\pi} < |z| < 1\} \setminus (0, 1) \exp[2\pi\theta i].$$

Moreover, φ_θ is everywhere locally invertible. The function $\tilde{u}_\theta = u \circ \varphi_\theta$ is harmonic and it satisfies all the assumptions from Theorem 2.3. By Theorem 2.3, \tilde{u}_θ has a continuous extension to the interval $(-1, 1)$ and this extension is 0 on that interval. By local invertibility of φ_θ this implies that the function u has a continuous extension to the closed unit disc \mathbb{D} and that $u|_{\partial\mathbb{D}} = 0$. The claim follows from the classical maximum principle. \square

A similar uniqueness principle holds for non-negative subharmonic functions. The proof is similar to that of Theorem 1.4, and is based on Theorem 2.1.

Theorem 5.1 (Uniqueness principle). *Let u be a non-negative subharmonic function on the unit disc. Assume that*

- (1) *there exist a lower semicontinuous function $f : \mathbb{D} \rightarrow \mathbb{R}_+$ and a measurable function $g : \mathbb{D} \rightarrow \mathbb{R}_+$ such that*
- (i) $u(z) \leq f(z)g(z)$ for every $z \in \mathbb{D}$,
 - (ii) $\lim_{\substack{z \rightarrow e^{i\varphi} \\ z \in \Omega_\theta(\varphi)}} f(z) = 0$ for every $\varphi \in [0, 2\pi)$ and some $\theta \in (0, \frac{\pi}{2})$,
 - (iii) $\sup_{r \in (0,1)} \int_0^{2\pi} g(re^{i\varphi}) d\varphi < \infty$,
- (2) *there exists $m \in (0, \frac{\pi}{\theta})$ such that*
- $$M_r(u) = \sup_{\varphi \in [0, 2\pi]} u(re^{i\varphi}) = O((1-r)^{-m}), \quad r \rightarrow 1-.$$

Then $u = 0$.

5.2. Analytic continuation across a linear boundary. In this section, we consider the square

$$R := \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1, -1 < \operatorname{Im} z < 1\},$$

and we study the question whether an analytic function $u : R \setminus \mathbb{R} \rightarrow \mathbb{C}$ admits an analytic continuation to the whole square R . For a thorough discussion of this type of problems see [6]. One of the analytic extension criteria is provided by the following classical *edge-of-the-wedge* theorem.

Proposition 5.2 (Edge-of-the-wedge). *Let $u : R \setminus \mathbb{R} \rightarrow \mathbb{C}$ be an analytic function. Assume that*

$$\lim_{\beta \rightarrow 0} (u(\cdot + i\beta) - u(\cdot - i\beta)) = 0$$

on $(-1, 1)$ in the sense of distributions. Then u admits an analytic extension to R .

Remark 5.3. Usually, the analytic extendability of u is derived from the existence and coincidence of the distributional limits $\lim_{\beta \rightarrow 0+} u(\cdot + i\beta)$ and $\lim_{\beta \rightarrow 0+} u(\cdot - i\beta)$. The (formally) more general Proposition 5.2 follows from [25, Theorem C] using Carleman's trick as in [32, Theorem A]; compare also with the proof of Theorem 5.4 below.

In particular, the conclusion of Proposition 5.2 holds if $\lim_{\beta \rightarrow 0} (u(\cdot + i\beta) - u(\cdot - i\beta)) = 0$ in L^p . However, if $\lim_{\beta \rightarrow 0} (u(\cdot + i\beta) - u(\cdot - i\beta)) = 0$ pointwise everywhere on $(-1, 1)$ then Proposition 5.2 can hardly be applied directly.

As a corollary to Theorem 2.3, we obtain the following edge-of-the-wedge theorem where distributional convergence is replaced by a combination of pointwise convergence everywhere and mean convergence. It improves the corresponding result in [9, Theorem 3.1].

Theorem 5.4. *Let $u : R \setminus \mathbb{R} \rightarrow \mathbb{C}$ be an analytic function. Assume that*

- (1) *there exist a lower semicontinuous function $f : R \setminus \mathbb{R} \rightarrow \mathbb{R}_+$ and a measurable function $g : R \setminus \mathbb{R} \rightarrow \mathbb{R}_+$ such that*
 - (i) $|u(z) - u(\bar{z})| \leq f(z)g(z)$ for every $z \in R \setminus \mathbb{R}$,
 - (ii) $\lim_{\substack{z \rightarrow \alpha \\ z \in \alpha + \Sigma_\theta}} f(z) = 0$ for every $\alpha \in (-1, 1)$ and some $\theta \in (0, \frac{\pi}{2})$,
 - (iii) $\sup_{\beta \in (0, 1)} \int_{-1}^1 g(\alpha + i\beta) d\alpha < \infty$.
- (2) *there exists $m \geq 0$ such that*

$$\sup_{\alpha \in [-1, 1]} |u(\alpha + i\beta) - u(\alpha - i\beta)| = O(\beta^{-m}), \quad \beta \rightarrow 0+,$$

Then the function u extends analytically to R .

Proof. The function $U(z) := (u(z) - u(\bar{z}))/2$ is harmonic in $R \setminus \mathbb{R}$. By the assumptions on u and Theorem 2.3, U extends continuously to the whole rectangle R and $U = 0$ on the interval $(-1, 1)$. We denote this extended function also by U . Since $U(z) = -U(\bar{z})$ and by the Schwartz reflection principle, U is harmonic on R .

There exists a harmonic conjugate V of U in R , so that $\tilde{u} := U + iV$ is analytic in R . The harmonic conjugate V is unique up to an additive constant.

The function $V_0(z) := (u(z) + u(\bar{z}))/2i$ is a harmonic conjugate of U in $R \setminus \mathbb{R}$, and since $U(z) + iV_0(z) = u(z)$, it follows that $\tilde{u} - u$ is constant in each component of $R \setminus \mathbb{R}$. Since V is continuous in R and V_0 is symmetric with respect to \mathbb{R} , we obtain that $\tilde{u} - u$ is constant in $R \setminus \mathbb{R}$. Thus, the function u extends analytically to R . \square

5.3. Stability of bounded C_0 -semigroups. Let X be a complex Banach space. Consider an abstract Cauchy problem

$$(5.1) \quad \begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in X, \end{cases}$$

where A is a closed linear operator on X with dense domain $D(A)$. It is a fundamental fact of the theory of C_0 -semigroups that (5.1) is well-posed if and only if A generates a C_0 -semigroup $(T(t))_{t \geq 0}$, [1, Theorem 3.1.12]. This semigroup is said to be *stable* if for every $x \in X$ we have

$$\lim_{t \rightarrow \infty} \|T(t)x\| = 0.$$

Recall that a function $u \in C(\mathbb{R}_+)$ is a *mild* solution of (5.1) if $\int_0^t u(s) ds \in D(A)$ and $A \int_0^t u(s) ds = u(t) - x$ for all $t \geq 0$. The importance of the concept of stability comes from the fact that the mild solutions of a well-posed abstract Cauchy problem (5.1) are precisely the orbits of the C_0 -semigroup $(T(t))_{t \geq 0}$ generated by A . Thus, in the terminology of differential equations, a stable C_0 -semigroup corresponds to a well-posed Cauchy problem (5.1) for which all mild solutions are asymptotically stable. One of the central problems in the theory of stability is to characterize stability of a semigroup in *a priori* terms of the generator, e.g. in terms of the spectrum of the generator or, more generally, its resolvent.

For accounts of (mostly spectral) stability theory, one may consult [1], [13], [24]. A discussion of recent developments can be found in [10]. In this section, we present several resolvent stability criteria and their “algebraic” counterparts. In a certain sense, these criteria are optimal.

By the uniform boundedness principle, a stable semigroup is automatically bounded, thus we will study stability of *bounded* semigroups in the rest of the paper.

The following stability criterion from [3, Theorem 3.1] (see also [28, Theorem 2.3]) will be fundamental for our study. Recall that a function $F : \mathbb{R} \rightarrow X$ is a *complete trajectory* for a semigroup $(T(t))_{t \geq 0}$ if $F(t + s) = T(t)F(s)$ for all $t \geq 0, s \in \mathbb{R}$. If $(T(t))_{t \geq 0}$ is the adjoint of a C_0 -semigroup, then F is weak* continuous on \mathbb{R} .

Theorem 5.5. *For a bounded C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X the following statements are equivalent:*

- (i) *The semigroup $(T(t))_{t \geq 0}$ is stable,*
- (ii) *The adjoint semigroup $(T(t)^*)_{t \geq 0}$ does not admit a bounded, nontrivial complete trajectory.*

Theorem 5.5 can be considered as a topological characterization of stability. To relate this characterization to analytic properties of the generator we will need the notions of Carleman transform and Carleman spectrum.

For every bounded weakly measurable (weak* measurable, if X is a dual space) function $F : \mathbb{R} \rightarrow X$ we define the *Carleman transform* \hat{F} by

$$\hat{F}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} F(t) dt, & \operatorname{Re} \lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} F(t) dt, & \operatorname{Re} \lambda < 0. \end{cases}$$

If $(T(t))_{t \geq 0}$ is a bounded C_0 -semigroup on X and $F(t) = T(t)x$ for $t \geq 0$, then

$$(5.2) \quad \hat{F}(\lambda) = R(\lambda, A), \quad \operatorname{Re} \lambda > 0,$$

where $R(\lambda, A)$ is the resolvent of the generator A of $(T(t))_{t \geq 0}$.

The Carleman transform \hat{F} is analytic in $\mathbb{C} \setminus i\mathbb{R}$. The set of singular points of \hat{F} on $i\mathbb{R}$, i.e. the set of all points near which \hat{F} does *not* admit an analytic extension, is called the *Carleman spectrum* of F .

The Carleman spectrum of a bounded function is nonempty unless the function is zero, see for example [1, Theorem 4.8.2]; indeed, the only entire function f satisfying $\|f(z)\| \leq c/|\operatorname{Re} z|$, $z \in \mathbb{C} \setminus i\mathbb{R}$, is 0. Thus, by the equivalence (i) \Leftrightarrow (ii) of Theorem 5.5, a bounded C_0 -semigroup is stable if and only if for every bounded complete trajectory F of the adjoint semigroup the Carleman transform \hat{F} extends to an entire function. This fact allows us to apply the analytic extension criterion from Theorem 5.4 to the study of stability of semigroups.

The following version of the resolvent identity for the Carleman transform of a complete trajectory appears to be useful for locating the Carleman spectrum of the trajectory, [4, Lemma 6.1].

Lemma 5.6. *Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a Banach space X with generator A , let F be a bounded complete trajectory for $(T(t)^*)_{t \geq 0}$, and let \hat{F} be its Carleman transform. Then for every $\lambda \in \mathbb{C}_+$ and every $\mu \in \mathbb{C} \setminus i\mathbb{R}$*

$$(5.3) \quad \hat{F}(\mu) = R(\lambda, A^*)F(0) + (\lambda - \mu)R(\lambda, A^*)\hat{F}(\mu).$$

In order to formulate our main results in this section, we need to recall that a Banach space X has *Fourier type* $p \in [1, 2]$, if the Fourier transform \mathcal{F} defined on the vector-valued Schwartz space $\mathcal{S}(\mathbb{R}; X)$ by

$$(\mathcal{F}\varphi)(\beta) := \int_{\mathbb{R}} e^{-i\beta t} \varphi(t) dt, \quad \beta \in \mathbb{R},$$

extends to a bounded linear operator from $L^p(\mathbb{R}; X)$ into $L^q(\mathbb{R}; X)$, $\frac{1}{p} + \frac{1}{q} = 1$, i.e. if the Hausdorff-Young inequality holds for X -valued L^p -functions. The class of Banach spaces with Fourier type greater than 1 is fairly large: it contains, for example, L^p -spaces with $p > 1$. The Hilbert spaces can be characterized by the property of having Fourier type 2. For a comprehensive survey of the theory of Fourier type one may consult [14].

The following result essentially improves the stability criterion in [9, Theorem 4.2].

Theorem 5.7. *Let A be the generator of a bounded C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X with Fourier type $p \in (1, 2]$. Assume that the set*

$$M := \{x \in X : \lim_{\alpha \rightarrow 0+} \alpha^{\frac{p-1}{p}} R(\alpha + i\beta, A)x = 0 \text{ for every } \beta \in \mathbb{R}\}$$

is dense in X . Then $(T(t))_{t \geq 0}$ is stable.

Remark 5.8. A partial case of Theorem 5.7 for $p = 2$ (i.e. when X is a Hilbert space) was proved in [27] by means of an involved operator-theoretic construction. The reasoning there does not extend to the more general situation considered here.

Proof. Let $F : \mathbb{R} \rightarrow X^*$ be a bounded complete trajectory for the adjoint semigroup $(T(t)^*)_{t \geq 0}$, i.e. a weak* continuous function satisfying $T(t)^*F(s) = F(t+s)$ for every $t \geq 0, s \in \mathbb{R}$. Let $x \in M$ and let $F_x := \langle F, x \rangle$. Let \hat{F}_x be the Carleman transform of F_x .

By Lemma 5.6 for every $\alpha > 0$ and every $\beta \in \mathbb{R}$,

$$\begin{aligned} |\hat{F}_x(\alpha + i\beta) - \hat{F}_x(-\alpha + i\beta)| &= |2\langle \alpha^{\frac{1}{p}} \hat{F}_x(-\alpha + i\beta), \alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x \rangle| \\ &\leq g(\alpha + i\beta) f(\alpha + i\beta), \end{aligned}$$

where

$$g(\alpha + i\beta) := \|2\alpha^{\frac{1}{p}} \hat{F}_x(-\alpha + i\beta)\|$$

and

$$f(\alpha + i\beta) := \|\alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\|.$$

By the boundedness of F and the Hausdorff-Young inequality,

$$(5.4) \quad \sup_{\alpha > 0} \|g(\alpha + i\cdot)\|_{L^q(\mathbb{R})} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Moreover, by (5.2),

$$\sup_{\alpha + i\beta \in \mathbb{C}_+} \alpha \|R(\alpha + i\beta, A)\| \leq \sup_{t \geq 0} \|T(t)\| < \infty.$$

By assumption $x \in M$, and the resolvent identity implies for every $\theta \in (0, \frac{\pi}{2})$ and every $\beta \in \mathbb{R}$ that

$$\begin{aligned}
& \limsup_{\substack{\alpha \rightarrow 0+ \\ |\beta' - \beta| \leq \alpha \tan \theta}} f(\alpha + i\beta') \\
& \leq \limsup_{\substack{\alpha \rightarrow 0+ \\ |\beta' - \beta| \leq \alpha \tan \theta}} \|\alpha^{\frac{1}{q}} (R(\alpha + i\beta', A) - R(\alpha + i\beta, A))x\| \\
& \leq \limsup_{\substack{\alpha \rightarrow 0+ \\ |\beta' - \beta| \leq \alpha \tan \theta}} \|\alpha \tan \theta R(\alpha + i\beta', A) \alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\| \\
& \leq \tan \theta \sup_{t \geq 0} \|T(t)\| \limsup_{\alpha \rightarrow 0+} \|\alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\| \\
& = 0.
\end{aligned}$$

It follows from this inequality, the boundedness of F_x and (5.4) that we can apply Theorem 5.4 in order to see that the Carleman transform \hat{F}_x extends analytically through the imaginary axis to an entire function. This implies $F_x = \langle F, x \rangle = 0$. Since M is dense in X , we conclude that $F = 0$, i.e. there is no nontrivial bounded complete trajectory for $(T(t)^*)_{t \geq 0}$. By Theorem 5.5, this is equivalent to the property that the semigroup $(T(t))_{t \geq 0}$ is stable. \square

Remark 5.9. Note that the conditions of Theorem 5.7 are not meaningful for $p = 1$. Indeed, $\lim_{\alpha \rightarrow 0+} R(\alpha + i\beta, A)x = 0$ implies $x = 0$ by the resolvent identity.

If A is the generator of a bounded C_0 -semigroup, then there is a variety of (equivalent) ways to define the *fractional powers* $(i\beta - A)^\gamma$ for every $\beta \in \mathbb{R}$ and every $\gamma > 0$; see, for instance, [23]. Note that $((i\beta - A)^\gamma)_{\gamma > 0}$ form a semigroup and so the ranges $\text{Rg } (i\beta - A)^\gamma$ satisfy

$$\text{Rg } (i\beta - A)^\nu \subset \text{Rg } (i\beta - A)^\gamma, \quad \beta \in \mathbb{R},$$

whenever $\nu > \gamma$. We proceed with a corollary of Theorem 5.7 which is a kind of algebraic criterion for stability. It can be considered as a limit case of Theorem 5.7.

Corollary 5.10. *Let A be the generator of a bounded C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X with Fourier type $p \in [1, 2]$. If*

$$(5.5) \quad \bigcap_{\beta \in \mathbb{R}} \text{Rg } (i\beta - A)^{\frac{1}{p}} \text{ is dense in } X,$$

then $(T(t))_{t \geq 0}$ is stable.

Remark 5.11. Corollary 5.10 improves the stability result from [9, Corollary 4.6]; the exponent $\frac{1}{p}$ in (5.5) was there replaced by an exponent $\gamma > \frac{1}{p}$.

Corollary 5.10 is optimal in the following sense: for every $p \in [1, 2]$ there exist a Banach space X having Fourier type p and a *nonstable* bounded semigroup with generator A such that for *every* $\gamma \in (0, \frac{1}{p})$

$$\bigcap_{\beta \in \mathbb{R}} \text{Rg}(i\beta - A)^\gamma \text{ is dense in } X;$$

see [8, Example 4.9].

Note that the stability condition from Corollary 5.10 is sufficient but in general not necessary for stability, except possibly in the case $p = 2$, [8, Example 4.1]. In fact, it is an open problem whether the condition (5.5) (or the stability condition in Theorem 5.7) characterizes stability in Banach spaces with Fourier type $p = 2$, i.e in Hilbert spaces.

Proof of Corollary 5.10. The case $p = 1$ has been proved in [4, Theorem 2.4]. The assumption implies that $\text{Rg}(i\beta - A)$ is dense in X . By the fractional mean ergodic theorem, [19, Proposition 2.3], [29, Proposition 2.2], for every $x \in X$ and every $p > 1$

$$\lim_{\alpha \rightarrow 0+} \alpha^{\frac{p-1}{p}} R(\alpha + i\beta, A)^{\frac{p-1}{p}} x = 0.$$

By [19, Theorem 2.4], [29, Theorem 2.4], for every $x \in \text{Rg}(i\beta - A)^{\frac{1}{p}}$

$$\lim_{\alpha \rightarrow 0+} R(\alpha + i\beta, A)^{\frac{1}{p}} x \text{ exists.}$$

Hence, if $x \in \text{Rg}(i\beta - A)^{\frac{1}{p}}$ and $p > 1$, then

$$\lim_{\alpha \rightarrow 0+} \alpha^{\frac{p-1}{p}} R(\alpha + i\beta, A)x = \lim_{\alpha \rightarrow 0+} \alpha^{\frac{p-1}{p}} R(\alpha + i\beta, A)^{\frac{p-1}{p}} R(\alpha + i\beta, A)^{\frac{1}{p}} x = 0.$$

The claim for $p > 1$ thus follows from Theorem 5.7. \square

Remark 5.12. We conclude with a remark concerning stability of discrete operator semigroups $(T^n)_{n \geq 0}$, where T is a bounded linear operator on X . *Stability* for a bounded linear operator T means that $\lim_{n \rightarrow \infty} \|T^n x\| = 0$ for every $x \in X$. All the above arguments can be directly transferred to the discrete case after an appropriate change of notions. Sketches of this procedure can be found in [9] and in [4]. We obtain the following analog of Theorem 5.7 in the discrete setting.

Theorem 5.13. *Let T be a power bounded linear operator on a Banach space X with Fourier type $p \in (1, 2]$. Assume that the set*

$$\{x \in X : \lim_{r \rightarrow 1+} \|(r-1)^{\frac{p-1}{p}} R(r\xi, T)x\| = 0 \text{ for every } \xi \in \mathbb{T}\}$$

is dense in X . Then T is stable.

Also Corollary 5.10 admits a discrete counterpart.

Theorem 5.14. *Let T be a power bounded linear operator on a Banach space X with Fourier type $p \in [1, 2]$. If*

$$\bigcap_{\xi \in \mathbb{T}} \operatorname{Rg}(\xi - T)^{\frac{1}{p}} \text{ is dense in } X,$$

then T is stable.

6. APPENDIX

In this Appendix, we show that Wolf's theorem (Theorem 1.3) is optimal with respect to the growth condition on $M_r(u)$.

Example 6.1. Given $\theta \in (0, \pi/2)$ and $\varepsilon_0 > 0$, there exists a function $u \neq 0$ harmonic in the unit disc such that

$$\begin{aligned} (i) \quad & \lim_{\substack{z \rightarrow e^{i\varphi} \\ z \in \Omega_\theta(\varphi)}} u(z) = 0 \text{ for every } \varphi \in [0, 2\pi); \\ (ii) \quad & M_r(u) = O(e^{\varepsilon_0(1-r)^{-\frac{\pi}{2\theta}}}) \quad \text{as } r \rightarrow 1-. \end{aligned}$$

For a similar construction see [7, Example 7.1].

Sketch of the proof. Choose small $\beta > 0$ and $\varepsilon > 0$. Let D_0 be the union of \mathbb{D} and a small neighborhood (in \mathbb{C}) of the arc

$$\{z \in \partial\mathbb{D} \setminus \{1\} : \frac{\pi}{2} - \theta - \frac{2\beta}{3} \leq \arg(1-z) \leq \frac{\pi}{2} - \theta - \frac{\beta}{3}\},$$

and let

$$\Delta = \{z \in \overline{D_0} \setminus \{1\} : \arg(1-z) > \frac{\pi}{2} - \theta - \beta\}.$$

We choose $K \in \mathbb{R}$ such that if $\arg(1-z) = \frac{\pi}{2} - \theta$, then $K + i\varepsilon \frac{1+z}{1-z} \in e^{i\theta}\mathbb{R}_+$. For large even A we define

$$f(z) = \begin{cases} \left(\frac{1-z}{1+z}\right)^A \exp\left(\left(K + i\varepsilon \frac{1+z}{1-z}\right)^{\pi/(2\theta)}\right) & \text{if } z \in \Delta, \\ 0 & \text{if } z \in \overline{D_0} \setminus \Delta. \end{cases}$$

Then $\operatorname{Im} f(z) = 0$ for every $z \in \Delta \cap \partial\mathbb{D}$, and $|f(z)| + |\nabla f(z)| = O(|1-z|^2)$ as $z \rightarrow 1$, $\frac{\pi}{2} - \theta - \beta < \arg(1-z) \leq \frac{\pi}{2} - \theta$.

Next, we fix $\tilde{g} \in C^2([-\pi/2, \pi/2])$ such that $\tilde{g}|[\frac{\pi}{2} - \theta - \frac{\beta}{3}, \frac{\pi}{2}] = 1$, $\tilde{g}|[-\frac{\pi}{2}, \frac{\pi}{2} - \theta - \frac{2\beta}{3}] = 0$, $0 \leq \tilde{g} \leq 1$, and we define $g(z) := \tilde{g}(\arg(1 - z))$, $z \in \overline{D_0} \setminus \{1\}$. Then $g \in C^2(\overline{D_0} \setminus \{1\})$ and

$$|\nabla g(z)| \leq \frac{C}{|1 - z|}, \quad z \in D_0.$$

Set $h := fg$. Then $h \in C^2(D_0) \cap C(\overline{D_0} \setminus \{1\})$, $\text{Im } h(z) = 0$ for every $z \in \partial\mathbb{D} \setminus \{1\}$,

$$|\bar{\partial}h(z)| = O(|1 - z|) \text{ as } z \rightarrow 1, \text{ and}$$

$$\lim_{\substack{z \rightarrow 1 \\ z \in \Omega_\theta(0)}} h(z) = 0.$$

Next, we set

$$h_1(z) = \frac{1}{2\pi} \int_{D_0} \frac{\bar{\partial}h(\zeta)}{z - \zeta} dm_2(\zeta).$$

Then $h_1 \in C^1(D_0) \cap C(\overline{\mathbb{D}})$ and $\bar{\partial}h_1 = \bar{\partial}h$ on D_0 . Therefore, the function $H = h - h_1$ is analytic in D_0 , $H \in C(\overline{\mathbb{D}} \setminus \{1\})$, and

$$\log |H(z)| \leq C + \frac{C\varepsilon}{|1 - z|^{\pi/(2\theta)}}, \quad z \in \mathbb{D}.$$

Let v be the Poisson extension of $\text{Im } h_1|_{\partial\mathbb{D}}$ to \mathbb{D} , and let $w := v + \text{Im } H$. Then w is harmonic in \mathbb{D} , $w \in C(\overline{\mathbb{D}} \setminus \{1\})$, $w = 0$ on $\partial\mathbb{D} \setminus \{1\}$, and

$$\begin{aligned} \log |w(z)| &\leq C + \frac{C\varepsilon}{|1 - z|^{\pi/(2\theta)}} \\ &\leq C + \frac{C\varepsilon}{(1 - |z|)^{\pi/(2\theta)}}, \quad z \in \mathbb{D}. \end{aligned}$$

Finally,

$$\lim_{\substack{z \rightarrow 1 \\ z \in \Omega_\theta(0)}} w(z) = \lim_{\substack{z \rightarrow 1 \\ z \in \Omega_\theta(0)}} (v(z) - \text{Im } h_1(z)) + \lim_{\substack{z \rightarrow 1 \\ z \in \Omega_\theta(0)}} \text{Im } h(z) = 0.$$

□

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